Computer SCIENCE and Mathematics in the Elementary Schools

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September 23, 1991

Abstract

Computer *science* is fundamentally about algorithms, recipes for solving problems and performing tasks. In the same way that children can learn about dinosaurs without digging for bones, and about planets and space without peering into telescopes, the intellectual core of computer science is not dependent on machines for its presentation. Just as with these other subjects, an approach based on stories, activities and ordinary materials can be more more vivid and engaging than approaches that make a fetish of computers. We argue that algorithmic topics are a good source of material with which to provide for children in the elementary grades a broad, exciting and active introduction to mathematics. Our experiences sharing some of these topics with classrooms in grades one through four (ages 5–9) are described. We propose that principles of language acquisition should be applied to the teaching of the mathematical sciences, and review how these principles have previously been applied to the teaching of reading and writing. We discuss some of the important aspects of the mathematics research community experience, and explore ways in which this experience can be fostered in the classroom. Some kinds of mathematical research in algorithms and combinatorics are actually accessible to elementary-age children, and conversely, interaction with children can sometimes inspire research questions. We describe some examples of this surprising research community.

1 Introduction

This paper describes and analyses the experiences of the author in presenting a variety of topics in computer science and discrete mathematics to elementary school children of ages five through nine. Some of these presentations were made with the collaboration of Nancy Casey, whose work is described elsewhere in this volume.

The discussion that is offered here can be summarized as follows.

1. The competencies required for the increasingly computerized world are essentially mathematical. It is a serious (and common) mistake to make a fetish of the machines.

2. Computer science is not about machines, in the same way that astronomy is not *about telescopes*. There is an essential unity of mathematics and computer science.

3. The intellectual core of computer science can be presented to children even in situations where there are no computers (for example, in countries or school systems that cannot afford them), laying a foundation for later computer science education. Many of the core ideas of computer science are best introduced without machines.

4. Computer science represents a tremendous flowering of mathematics. It is particularly good news for children because it is a treasury of accessible, colorful and active mathematics. For introducing children to mathematical science, it is unmatched in these terms by any other source. Think of computer science as the modern "geometry," but a thousand times more vivid, varied, engaging, and open to exploration.

5. The teaching of the mathematical sciences should follow the lead of, and be integrated with, the "whole language" paradigm in the teaching of language and writing skills. Mathematics that is rich with stories and opportunities for active exploration is well suited to this language-acquisition point of view.

6. In the same way that children's art is interesting *as art* and children's writing is interesting as writing, mathematics with children can be interesting as mathematics. There are kinds of research activity accessible to children, and interaction with children can be stimulating for people active in research or at higher levels of learning.

The organization of this paper roughly follows the story as it unfolded, first presenting material in the classrooms, and later attempting to gain a broader understanding of how these experiences fit into the larger field of current discussions in mathematics education. The classroom experiences came first, because my involvement began simply as a "parent volunteer" contributing classroom topics for an hour or an afternoon.

Section 2 describes (retrospectively) the objectives of these classroom presentations. Section 3 provides some details of the topics and activities which were brought to the classrooms. Section 4 concerns the subsequent effort to relate these classroom experiences to current discussions in the field of mathematics education. Section 6 describes a supportive point of view for these activities in language arts education. Section 7 discusses how classrooms can function as research communities. In section 8, some mathematical research problems that were inspired by the classroom presentations are recounted.

2 The Objectives of Our Classroom Projects

There seem to be at least two fundamental problems with education in the mathematical sciences in grades 1-4.

- Most children in these grades are never exposed to mathematics. Arithmetic is not mathematics!
- Most children in these grades are never exposed to computer science, despite all the PC's in the classrooms. Programming is not computer science!

In contrast, children in these grades are often exposed to the central questions and activities of geology, astronomy, biology, chemistry, etc. They are sometimes exposed to the frontiers of knowledge in these subjects, as exciting recent discoveries and developments are discussed in class. They are exposed to art, music and literature, and their creative efforts in all of these areas — their writings, art and science projects are valued.

There is a tendency to apply miserly (and mistaken) standards of "real world" concern to the curriculum of the mathematical sciences that are not applied to any other subject. It is not approached as the playful, fascinating and beautiful enterprise that it is, competitive with dinosaurs and outer space. It is usually treated rather as the necessary dreary accumulation of skills for someday balancing checkbooks and figuring mortgages. What life-skill needs do the subjects of dinosaurs and outer space address? If the exposure of children to literature were similarly limited to tax forms, job applications and parking regulations, then reading would be as widely loved as mathematics is today.

Mathematics, the language of science, and its principal modern branch, computer science, can be presented to children in these grades in wonderfully engaging and active ways, emphasizing their role as the language of science and technology. Children can be presented from the beginning with the essential unity of mathematics and computer science. Mathematics presented as a research enterprise can also provide fair opportunities for children to be shown that the world is full of questions to which adults do not know the answers.

The central questions of computer science are conceptual, and appreciating this science does not depend on sitting in front of a terminal, any more than appreciating the questions of astronomy depends on holding your eye to a telescope. Similarly, the competencies that are most important for coping with an increasingly computerized world are essentially mathematical. Programming and "experience with computers," are relatively unimportant in contrast to mathematical literacy, and confidence in mathematical modeling and problem-solving.

In many school situations where there is not a machine for every student and there are scarce opportunities for using the machines, making a fetish of computers may have the further negative effect of increasing the disadvantage of female and minority students who tend to lose these opportunities to pushier cohorts.

Computer science is thoroughly permeated with discrete mathematics. Together these subjects constitute a fertile source of accessible, colorful and concrete problems for presenting mathematical modeling, reasoning and open-ended exploration. For the early school years these subjects are unmatched in this regard by any other kind of mathematics. Computer science adds to this richness, as it does to mathematics in general, by highlighting and elaborating the issues of computational activity and resource economy. These intertwined subjects constitute a modern treasury of accessible, active and applicable mathematics.

The goals of the visits to elementary school classrooms can be summarized as follows:

1. To show that mathematics is fun and full of stories, activity, invention and play.
2. To show that mathematics, like dinosaurs and outer space, is a live science with visible frontiers of knowledge.
3. To present the essential unity of mathematics and computer science and display the intellectual core of the latter.

Working with young children is interesting for several possible reasons. First, lasting attitudes towards mathematical science may be formed in the earliest grades. Secondly, competition with the deadly traditional "school mathematics" curriculum is less of an issue in these grades, and the deficiencies of the traditional curriculum are more starkly apparent. Thirdly, if interesting "college level" topics (such as the Muddy City problem described in the next section) can be made accessible and interesting to second-graders, then these topics will also be accessible at intermediate levels.

The outlook of the author is that of an active researcher in mathematics and computer science, and the story that is recounted in this paper is basically that of an enthusiastic, intellectually naive adventurer in the world of elementary mathematics education.

3 Some Math and Computer Science Topics for Young Children

The purpose of this section is to describe some of the topics which were presented during the classroom visits. These visits were to middle-class classrooms of children aged 6-10 in Moscow, Idaho, and Victoria, British Columbia during the period of time September 1989 to June 1992. The typical format for a presentation was a 60 minute block of time in which to present the topic and organize activities and discussion. In one of the classrooms (second grade) the children kept individual mathematics journals.

3.1 The Muddy City

The technical name for this topic is the problem of computing a *minimum weight spanning tree* in a graph. Several efficient algorithms for solving this algorithmic problem are known and are routinely covered in any college level course on algorithms and complexity [CLR]. The story presentation of the problem next described is meant to be entertaining, but it should be noted that there are many practical applications of this problem in computing. This is true of many combinatorial optimization problems — they support the modeling of both fanciful and industrial situations.

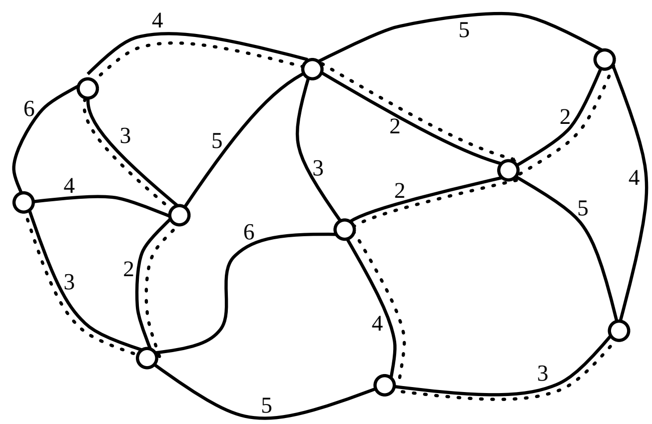

Figure 1: The Muddy City and an optimal solution. Citizens of Muddy City want to pave enough roads so that anyone can get anywhere. They want to keep the cost down, and don't mind if the traveling distances on the paved roads are long.

The children are given a map of the Muddy City (See Figure 1) and told the story of its woes — cars disappearing into the mud after rainstorms, etc. The mayor insists that some of the streets must be paved, and poses the following problem. (1) Enough streets must be paved so that it is possible for everyone to travel from his or her house to anyone else's house by a route consisting only of paved roads, but (2) the paving should be accomplished at a minimum total cost, so that there will be funds remaining to build the town swimming pool. The number associated with each street in the map of the town represents the cost of paving that particular street.

Thus the problem posed is to devise a paving scheme meeting requirement (1), connecting up the town by a network of paved roads, that involves a minimum total ammount of paving. The cost of a paving scheme is calculated by summing the paving costs of the roads chosen for surfacing.

Some of the 5-year-olds began by figuring out where the new town swimming pool should be located, and which node represented their house! In general, posed with a problem of this sort, the classroom explodes with activity, and there is a tremendous range of response. Some students rapidly understand the problem, while others require further explanation as they consider partial solutions and examples. (It is always useful to delegate to those who understand the problem the job of explaining it to others as tutors.) One fascinating aspect of the classroom experience was the reports of the teachers that their expectations concerning student performance were turned topsy-turvy: the children who did well on these problems were not always the same as the ones who did well at the usual arithmetic drill.

The children worked on the problem, usually in small groups, with the immediate objective of finding the best possible solution. This was typically recorded in a place that everyone could see. Students were asked to describe their strategies and ideas, both as they worked and in a concluding discussion. In classrooms where the students kept mathematics journals, they also wrote descriptions of the problem and of their ideas on how to solve it. These math journals were instituted with great success in a (latter part of the year) second-grade classroom, and in a fourth-grade classroom.

As part of the wrap-up discussion, we sometimes presented Kruskal's algorithm (one of several known algorithms for solving this problem efficiently). This method of finding an optimal solution consists simply of repeatedly paving a shortest street which does not form a cycle of paved streets, until no further paving is required. It is interesting that the children often discovered some of the essential elements of Kruskal's algorithm and could offer arguments supporting them. (Rediscovering Kruskal's algorithm is not the point, of course.)

The natural questions that turn up in discussing this problem are rich and varied, and include such matters as, "How can you quickly tell if a proposed paving scheme meets requirement (1)?" "How can one determine if a solution can be improved?" "What is the minimum *number* of streets paved in an optimal solution?" A variety of interesting observations can (and will) be made.

In projects like this, it is not important that the teacher anticipate in advance the nuances of the problem-solving discussion that will be generated. What is important is that the children are presented with a plausible and engaging story-problem, which provides a rich field of play for common-sense mathematical reasoning. What is important for the teacher to do after the problem has been posed is to encourage and facilitate invention and discussion. (Knowing Kruskal's algorithm and other background material is not important.) When children work on problems of this sort, a rich structure of observation, argument and solution strategies will *always* emerge.

This problem can be presented to classrooms of children aged 5–6 by using maps with distances marked by ticks rather than numerals, so that the total amount of paving can be figured by counting rather than by sums.

3.2 Map Coloring

Maps are passed out and it is explained that in map coloring, two countries which share a border (such as Canada and the United States) should be colored with different colors. The story concerns the poor map-colorer, trying to eke out a living with just a few crayons. How hard is it to tell whether two colors are enough for a given map? (There is an easy way to answer this.) How difficult is it to tell whether three colors are enough? (No one knows an easy method, and indeed the seemingly innocuous question of whether there is an easy procedure

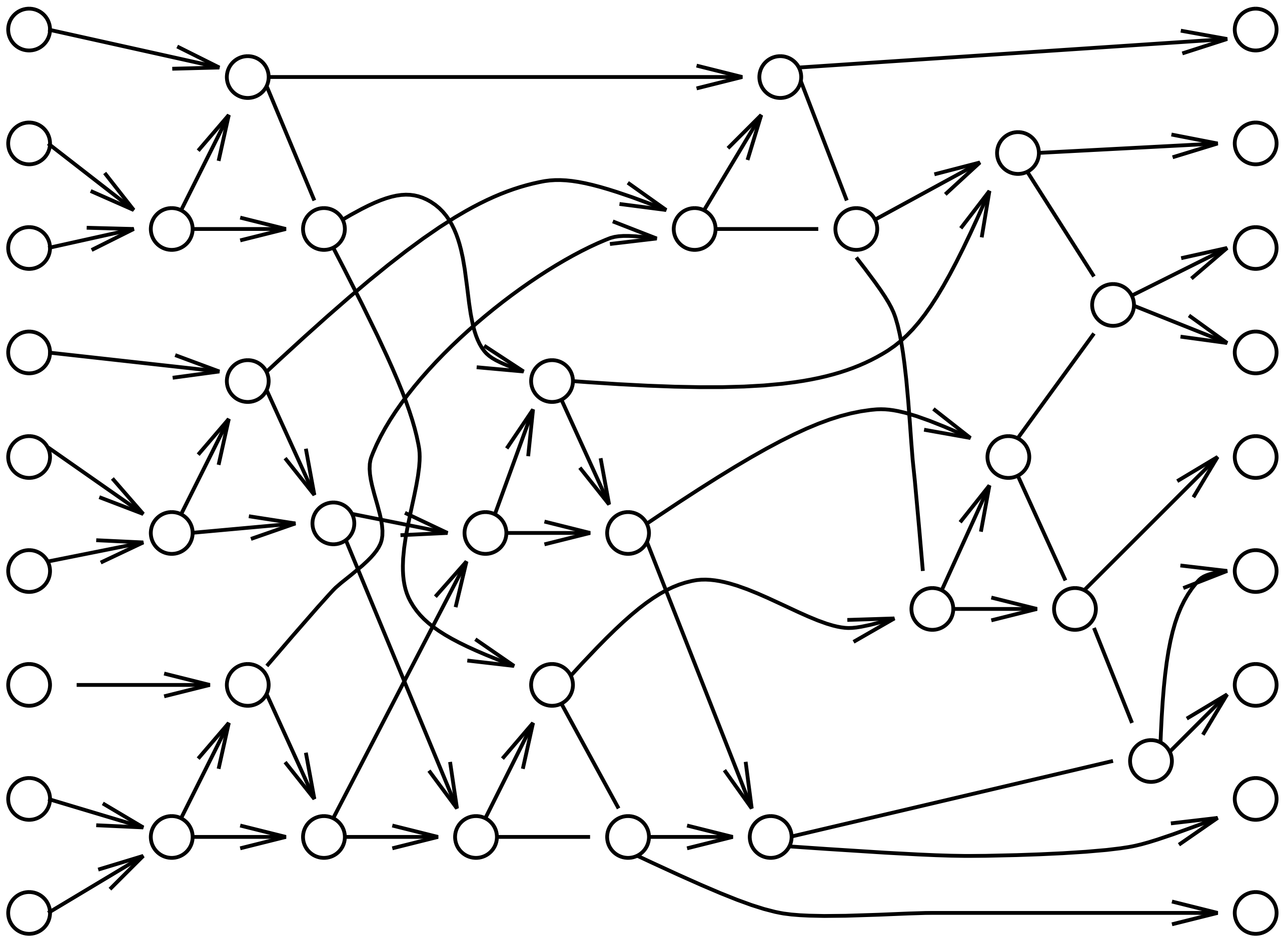

Figure 2: An example of a sorting network.

for finding the answer is equivalent to what is widely regarded as the most important open problem in computer science and the foundations of mathematics [GJ].)

The best solutions found for the maps under consideration (that is, the solutions using the smallest number of colors) can be displayed as attempts are made to improve these solutions. Is there a way to tell if a solution cannot be improved?

This problem is extremely rich with possible strategies, observations and ideas. For example, one idea that often turns up is to use one color on as many countries as possible before beginning to use another color. The fact that four colors are always enough (the Four Color Theorem) was occasionally discussed.

3.3 Sorting Networks

A sorting network, an example of which is shown in Figure 2 , has n input lines and n output lines. Each *comparator node* of the network has two input lines and two output lines and functions in the following way: Regardless of how a pair of values arrive on the input lines to

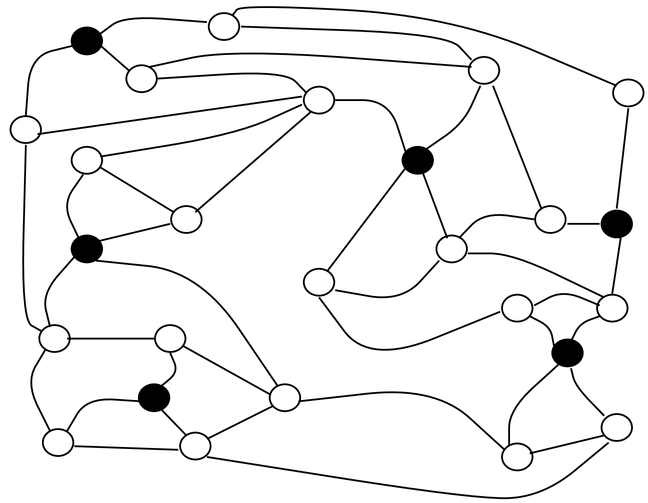

Figure 3: Sixtown and a minimal solution.

a comparator node, the largest value exits on the bottom output line and the smallest value exits on the top output line. The network has the property that however a set of n values is supplied to the input lines, the values will emerge in sorted order (increasing from top to bottom) on the output lines.

Groups of children were given the project of building sorting networks from pictures in a book [Kn]. The networks were constructed with colored tape on a large area of linoleum floor. (On another occasion the construction was outside in a paved area using colored chalk. A network having 10 inputs is of reasonable size.) The network was "operated" by having ten children walk through it, carrying values (numbers or words for alphabetical ordering) on slips of paper. On following a line to a comparator node (marked by a circle) a carrier waits until another carrier arrives on the other input line to the node; they then compare their values and decide which exit lines to follow as they continue through the network.

3.4 Ice Cream Stands and Firestations

This problem is known in the mathematics and computer science literature as the problem of finding a *minimum dominating set* in a graph [BM,GJ]. This has many practical applications, and is a classic computational problem of computer science.

The story goes like this. In order to prepare Sixtown for summer, we decided to build ice cream stands on various corners so that from any corner in the city one could reach an icecream stand by walking at most one block. We wished to be efficient, and the problem was to find a solution using a minimum number of stands. See Figure 3 for a map of Sixtown and a solution requiring only six stands.

On another occasion the same problem was posed with a story concerning the placement of firestations. It is, of course, not necessary to know what the minimum number required for a given map *is* in order to pose this problem. Many interesting observations and approaches will *always* emerge when this problem is worked on.

There are several interesting wrinkles to this problem. The author created the map of Sixtown by *beginning* with a much simpler figure for which the solution shown in Figure 3 is obvious, and then adding further "disguising" lines. By working in this way, it is possible to create town maps for which one (privately) knows a very efficient solution (because it has been "built in"), but for which it is often very difficult for someone else to find any solution equally efficient. This is an example of the beautiful and fundamental concept of a *one way function*, one of the conceptual building blocks of modern cryptography. On several occasions this topic was explored further. The children were charmed by the idea of creating maps which stymied their parents but for which they knew a secret solution.

There are literally dozens of graph-theoretic concepts such as dominating sets, for which stories can be invented or taken from the scientific literature, to create a rich playground for mathematical exploration and invention (see [BM,GJ,Ro]). In the mathematical literature concepts are often first introduced with a real or imaginary story of an application. One of the important aesthetic criteria at work in the mathematical sciences, especially computer science, is that an interesting concept is one that has an interesting story. If you can invent a mathematical problem with a good story, you have invented a problem worth exploring. There are many beautiful mathematical concepts yet to be invented by acts of story-telling.

3.5 Popsicle Stick Exploders

This is not a classical topic, but it will serve as an example of the kind of mathematical problem which anyone could recognize or invent. In Figure 4 is shown a diagram of one possible construction of an exploder using six sticks. When the sticks are woven together as indicated, the tension from the flexing of the sticks renders the ensemble stable. If the structure is thrown with a small amount of force against a wall, it explodes with sticks flying in all directions as the flexing tension is released.

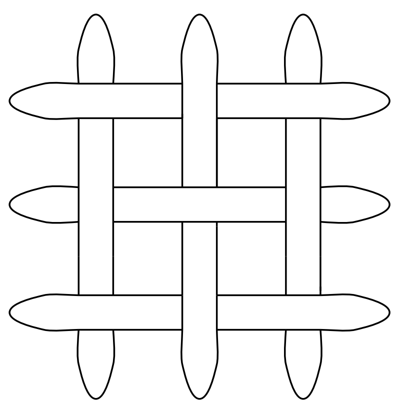

Figure 4: A popsicle stick exploder.

Natural questions for exploration include, "What is the minimum number of sticks with which you can construct an exploder?" "How many different exploders can be constructed with n sticks?" "Is it possible to construct arbitrarily large exploders?" It would be entirely possible for a class of 9 or 10 year olds to investigate a topic like this and publish their findings in a mathematics journal.

There is an amusing anecdote which goes with this topic (and there are similar anecdotes being held in reserve about "staging problems" with several of these topics). In presenting this topic to a group of children aged 4–7, the construction shown in Figure 4 was first demonstrated, with the intension that the children would then experiment with constructions. The children were extremely eager, of course, as they love all things that explode. Within fifteen minutes, however, over half the class had been reduced to tears of frustration! Unanticipated in posing this problem was the amount of hand strength and dexterity required to assemble the constructions. (This problem can be alleviated by using thinner sticks.)

3.6 Knots

For all the fuss in the mathematics education literature over mystified notions such as *the concept of number* and (relevant to this topic) *spatial sense*, it is amazing that such an illuminating and beautiful geometric topic as knot theory is invariably neglected. And this while children are *playing* in the schoolyard with braids and cat's cradle! Knots, something that everyone in the world uses, have a mathematical theory that figures significantly in intellectual current events in physics, chemistry, pure mathematics and biology [St].

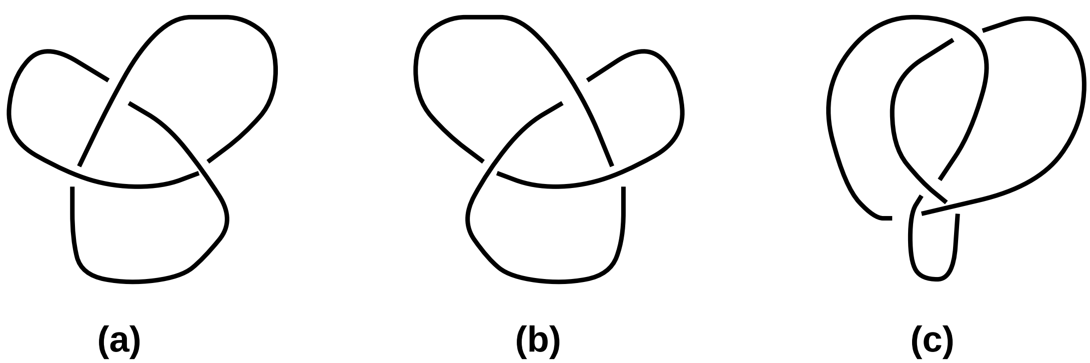

Figure 5: Some knots. (a) and (b) are mirror images of each other. (c) is the mirror image of itself.

Knots need no special introduction, only the explanation that for the mathematics of knots, the ends of the rope are joined. This topic is obviously fun to explore with lengths of cord and tape for joining the ends. There are a variety of intriguing questions that can be explored manipulatively. For example, is it possible to turn the knot of Figure 5(a) into the knot of Figure 5(b)? (The answer is, "no.") The knot depicted in Figure 5(c), however, is the *same knot* as its mirror image.

Fascinating theorems that can be explored by manipulation include the No-Unknotting Theorem (see Martin Gardiner [Ga]), and the result that every knot can be put into the form of a closed circular braid. Both of these theorems have an element of the mathematical quality of surprise, and therein lies the charm of trying them out through manipulation. Apart from any theorems, there are many ways to play inventively and mathematically with knots, formulating and exploring challenging questions (some of which are presently significant open problems).

3.7 Optimal Small Network Constructions

This is the Age of Networks, and there are many vital applications in modern technological systems of many kinds of small network designs optimizing a variety of network properties. Figure 6, for example, shows the largest known planar graphs for a few values of the parameters *maximum degree* and *diameter*. The *maximum degree* of a network is the maximum number of lines incident with any node. The *diameter* is the maximum distance (counting the number of lines to be traversed in a shortest route) between any two nodes in the network. Thus, the diameter measures the maximum number of times that a message sent through the network might need to be relayed.

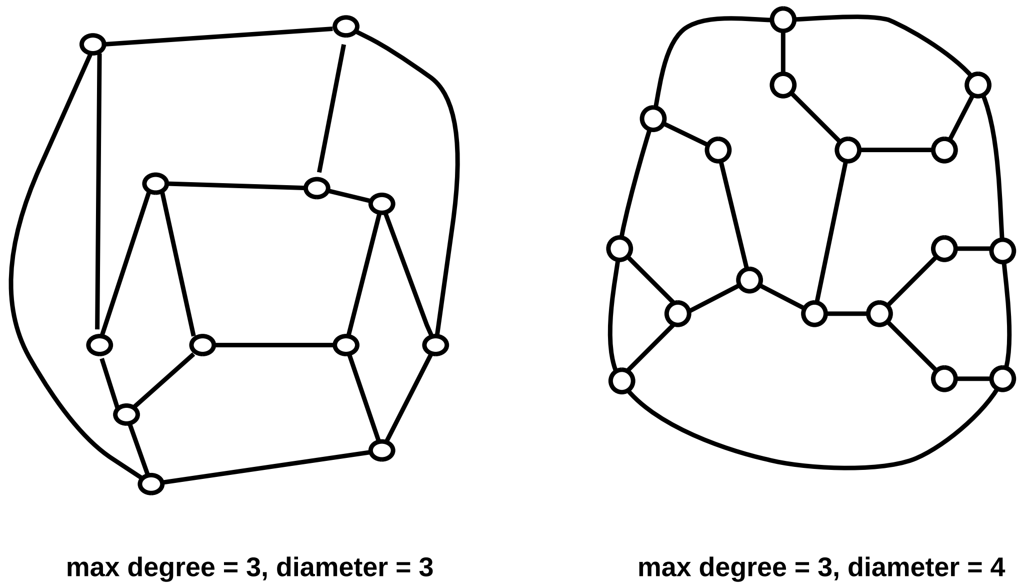

Figure 6: Degree/diameter constructions. These are the largest known planar graphs with the values indicated for the maximum degree and diameter.

It is not known whether or not there are larger networks than those shown for the indicated parameter values. This is a good example of a kind of mathematical problem on which children could actually do research nearly as well as trained mathematicians. The reason is that training is of essentially no help for this sort of problem where the combinatorial object is small. If larger networks exist, they will be found by paper, pencil, intuition, and experiment — "no experience or background required." There are many problems of this sort involving tradeoffs of various parameters, having applications in many different kinds of network engineering.

One of the classroom visits began with a description of this research problem and an offer of a reward consisting of a lunch date and a trip to the bookstore for anyone finding a larger construction than the largest ones presently known. A model of the maximum degree 3 and diameter 3 network shown in Figure 6 was built on the floor using colored tape, and the students hopped around on this, experiencing first hand that it did indeed have diameter 3. Note that this is an interesting and potentially valuable encounter with logical quantification, since the diameter property of the network is that *for every* pair of nodes, *there exists* a route between them of length at most 3.

The children were charmed and excited by the information that reward offers for the solution of mathematical problems is a part of the mathematics culture, and by my stories about a certain famous elderly mathematician who has made many such reward offers. An impressive amount of work on this problem continued for several weeks.

3.8 Other Topics

The few project areas described above hardly scratch the surface of the treasury of accessible and engaging topics in computer science and combinatorial mathematics. Sorting algorithms (sequential, parallel, randomized, on networks — there are many varieties) can provide much amusement and food for thought, and opportunities for game-like physical activity [Kn,Kr]. The puzzle books of Smullyan are a wonderful source of mathematical and logical riddles [Sm]. We experimented with randomized algorithms to decide Who Pays For The Tea. We enacted the Game Show Problem (recently popularized by the radio show Car Talk and an article in Parade Magazine), keeping statistics as each student played the role of the contestant. We played Search Number and other games on graphs [GJ]. We planned the route of the Traveling Salesman [GJ], devised One-Way Street Assignments [Ro], Cut Stock to build the doghouse [GJ] ... and we'd still hardly scratched the surface.

4 The Search for Legitimacy in Mathematics Education

This part of the story concerns explorations in the rather foreign (to this researcher) world of mathematics education, seeking some justification and sympathetic connections for the exciting and rewarding classroom experience recounted in the last section. Two impressions concerning this experience were foremost:

1. The enthusiasm of the children and the teachers.

   A typical conversation with a child busily trying to color a map with the minimum number of colors, would go something like:

   > "This is really fun!"
   > "Yes, this is a fun kind of mathematics, isn't it?"
   > "THIS is mathematics!? This is mega-mathematics!"

Several teachers stated that their picture of mathematics had been, "changed forever."

2. The enthusiastic support of the mathematics and computer science research communities.

As an example of the latter, when the motion to stamp a Committee on Education to assemble the SIGACT Compendium of Theoretical Computer Science for Children was offered at the business meeting at STOC '92 in Victoria, it was passed unanimously and enthusiastically [Ro]. This is entirely consistent with the experiences of the author whenever the intellectual possibilities for children and experiences such as described in the last section have been discussed informally at research meetings.

In contrast to the encouragement received from the classroom and research communities, initial feedback from the world of mathematics education was notably *discouraging*! For example, an official of the NCTM responded to the enthusiastic tale of the classroom experiences described in the last section with the statement, "Well, I hear that *you* are having a lot of fun, but how do you know that what you are presenting to the children is *balanced*? What is your *organizer*?"

The major differences between the approach embodied in the program of classroom experiences described in the last section and what seems to be the prevailing discussion in the world of mathematics education can be summarized:

1. With the exception of the Berkeley Family Math program [Er,STC], the word "fun" and the spirit of intellectual community and excitement seems difficult to find in discussions of mathematics education and its possible reform. To a researcher this seems to be a significant omission in a world that overwhelmingly relates to mathematics with fear and loathing, while the mathematicians are having so much fun.

2. One of the basic goals of the classroom experiences described in this paper is to bring to young children engaging, active, open-ended, story-full mathematics topics. The selection principle is basically, "Anything goes." Anything that can be made accessible enough to be interesting to the kids, especially if it stretches their experience of the mathematical with objects, questions, and problems they have not yet encountered. In contrast, virtually all discussion of content curriculum in the mathematics education world is enslaved to hierarchically organized conceptual bus schedules embodying pessimistic assumptions about children's emerging abilities and interests, and inexcusably static and narrow assumptions about the nature of mathematics. [1]

3. Another goal of the classroom experiences described in this paper is to bring to young children an appreciation of the frontiers of human knowledge — an appreciation of some of the mathematical questions to which no one presently knows an answer. We intend to compete with dinosaurs and outer space! Discussion of this objective appears (to the

[1] There is a body of recent literature that we would characterize as "good rap, bad examples," that talks about creating something resembling a research environment in the classroom, and then falls back for illustration on the usual limited and boring material [DMN,NCTM].

author, admittedly an outsider) to be essentially absent from the mathematics education literature.

4. Nowhere in the mathematics education literature does there seem to be a discussion of the possibility of real research projects that can be actively pursued by children.

In addition to the above qualitative differences between the classroom experiences described in this paper and the main points of view articulated in the mathematics education literature, there seems to be a remarkable *timidity* even in the best of that literature. For example, consider the suggestion that an appropriate topic to present "algorithmic thinking" to the "pre-algebra" age group would be to have them write down a detailed list of instructions for placing a long distance phonecall [Do]. That would be about par for the excitement level all the world has come to expect of school mathematics.

Most of the mathematics education literature shows little or no awareness of the tremendous developments going on in mathematics and its modern applied branches, developments that are changing our picture of what mathematics *is*, and its role in human affairs.

In trying to understand the strangeness of the educational world and the peculiar and definitely negative role of the mathematical sciences in that realm for most people, it seemed reasonable to put forward the following hypotheses for further investigation.

5 A Paranoid Theory of Mathematics Education

It is sometimes offered that the the importance of the traditional school mathematics curriculum is to teach children the "discipline of thinking." Yet it seems far more likely that the traditional curriculum serves an abusive hidden agenda contrary to the development of critical intellect and the spirit of inquiry and problem-solving. This hidden agenda may include:

(i) that the ruling social classes and authority structures do not prefer an inquiring and numerate public confident in its problem-solving ability,

(ii) that school mathematics provides excellent training in the obeying of arbitrary obscure procedures in the context of penalizing supervision,

(iii) that school mathematics provides a model for mystery cloaking the power of authority and can be effectively used to instill a sense of inferiority and self-blame on students, and

(iv) that ability in school mathematics provides a convenient rationalization for sorting children into opportunity tracks by social class or race (as ability in Latin once was used).

What if the traditional mathematics curriculum is not really about *mathematics*, but rather, in some large measure, about *authority, power, and social rationalization*? I think the question needs to be raised. The paranoid theory, whatever its ultimate merits, has had an interesting life. At the MER workshop, the author was asked to wait in the middle of the talk while a number of people copied down the paranoid theory. Privately, several prominent mathematicians have offered the opinion that there is considerable truth in it.

6 Teaching Mathematics as a Language

Eventually, a point of view in educational theory was located that seems to be sympathetic to the kinds of classroom experiences described in Section 3. This perspective was found not in the mathematics education literature, but rather (surprisingly?) in the literature and community of education in the language arts. In this section we describe this point of view and how it can be applied to the teaching of mathematics, "the language of science."

Recent years have seen a profound shift of perspective in the language arts education community towards a point of view that is sometimes termed *whole language*. Rather than a specific set of practices, this is a perspective on language acquisition that has classroom implications extending far beyond literacy [AEF].

6.1 The Whole Language Perspective

The whole language point of view has its roots in a large body of recent research in linguistics and cognitive psychology on language acquisition [Go1,Goll]. The central fact to which this research points is that children acquire language through actually using it in a community of language users, not through practicing its separate parts until some later date when the parts are assembled and the totality is finally used. Language competency develops in a child in a way that does not depend upon instruction and drill; this fact is a central reference point in modern linguistics.

The whole language perspective based on research in linguistics and cognitive science can be summarized in its essentials as follows:

1. The model of acquisition through real use (not practice exercises) is the best model for thinking about and assisting with all forms of language learning and learning in general.

2. Language competency is a complex interactive system with many parts (purpose and pragmatics, syntax and semantics of cuing systems, social context, etc.) and it is not reducible to those parts.

3. The development of language competencies in a child is seen as unfolding naturally and incidentally when that language is a part of the functioning of a community.

4. Whole language classrooms seek to provide a richly varied and engaging environment of real language usage. The class is a community of language users, and the task of the

teacher is to monitor and assist individual students in their projects, to diagnose any "stuck points," and to encourage the competencies presently under construction by the individual.

For lengthier descriptions of the whole language perspective on reading and writing, see [Go1,Go2,Goll,Ne]. A few works for teachers relating the whole language perspective to mathematics education have recently appeared [BB,BSS].

6.2 A Second Look at Our Project

From the whole language perspective, the perspective embodied in the classroom visits described in this paper suddenly makes sense! The primary concern is not with a scheduled hierarchy of skills, but rather with providing a mathematically rich environment, utilizing whatever interesting material is handy.

The exercise of routine skills, such as addition, was incidental to problem-solving. For example, in the Muddy City problem, the lengths of the paved streets must be tallied.

The goal of presenting visible frontiers of knowledge can be viewed in the whole language perspective as part of constituting the classroom as a community of mathematical language users, and as welcoming them to the larger community of mathematical literacy. The open-ended and exploratory nature of the mathematical topics and projects with the children made the classroom a research community. The functioning of this community while working on a topic involved a rich mix of verbal, written, social and thinking activities.

6.3 Mathematical Thinking and Language

In a whole language classroom, the context for real reading and writing is often supplied by other subjects [AEF]. The enrichment of childrens' mathematical environment by supplying a wide-ranging experience of collective mathematical problem-solving in a classroom which functions as a mathematical research community can provide valuable opportunities for exercising and sharpening important kinds of language skills. Having an engaging playground of opportunities for the kinds of language tasks involved in articulating precise questions, presenting and justifying logical reasoning, etc., has obvious value for written and oral language mastery. In the end, a whole mathematics curriculum may strengthen a whole language curriculum.

7 The Classroom Research Community

It can be seen that the easiest way to summarize the project described in this paper, is that it attempts to convey to elementary school classrooms the experience of participating in the mathematical science research community. The idea that the classroom should function as a literate community of readers and writers is central to the whole language approach to the teaching of language skills. The project described in this paper is thus fundamentally in tune with this outlook.

What aspects of the mathematical science research community experience are portable to elementary school classrooms? Some vital parts of that experience most definitely *can* be brought to the elementary grades.

7.1 Playfulness

It is easy for a practicing scientist to share this part of the experience of participating in the research community with children in the earliest grades, because young children still know vividly how to play, and have some natural solidarity as a research community.

Kurt Vonnegut pointed to this commonality in an amusing way when he remarked, "If you are going to teach, you should either teach graduate school or fourth grade. ... And if you can't explain it to fourth graders, you probably don't know what you're talking about."

Playfulness is related to abstraction and modeling. Presented with a map of the Muddy City, young children are quite comfortable with regarding the dots as representing houses, etc. (After all, they just finished asking their parents to regard an odd bit of stick or a paper cut-out as a laser gun.) College students are more likely to complain about a perceived lack of "realism" in the model.

Playfulness is often deeper than it appears. For example, from a "serious" perspective, the problem of the map-colorer eking a living with a few crayons may seem to be a fairy-tale, and a silly waste of time, compared to "real" and "practical" school mathematics such as the mechanics of long division. This fairy-tale problem is both mathematically profound, and has many important industrial applications. Do not forget Einstein's famous advice about what physics texts are most important to read: "Fairy tales, and more fairy tales." It is the sense that mathematics permeates the world in this way, that informs the aesthetic in the research community that loves a concept that has (or makes) a good story.

7.2 Asking Questions

Of course, the research community is always asking and trying to answer questions. In the mathematics research community, a good deal of recognition can come from just asking a good question, quite apart from being able to answer it. Question-asking can be made an important part of the classroom mathematics community; there is essentially no place for it in the traditional school mathematics curriculum with its hopelessly narrow and petrified view of mathematics.

Associated naturally with the importance of question-asking, is that it's OK not to know the answers. This makes for a radical shift in the relationship between the teacher and the class in the contrasting settings of traditional and whole mathematics classrooms. To participate in the latter, teachers must see that creating an interesting problem-solving environment where children can ask questions for which the teacher doesn't know the answer, is *positively* a good thing. It is *not* necessary, or even desirable, to know the answer (or the "background") before posing the story-problem.

7.3 A Complex Relationship to Truth

In the traditional school mathematics classroom, truth plays a strikingly simple role, in stark contrast to both everyday problem-solving of any kind, and to the complex experience of truth in the mathematics research community. Mathematical statements as they occur in the research community are richly colored in a variety of important ways that tend not to be well-appreciated presently outside of that community. A statement may be intuitively clear and have an easy proof, or it may be strongly intuitive (for example, that the first two knots in Figure 5 are not equivalent) but have only a difficult proof. A statement may be intuitively clear, but have no known proof. And it can sometimes turn out that intuition is wrong! Good mathematics requires both unrelenting skepticism, and wild imagination.

Much could be said about the humanizing value of a rich experience of truth. This experience can be encouraged in classrooms where children formulate, discuss and explore conjectures. These may concern such questions as, "How big is the moon?" or "How many berries are in the berry patch?" There are also beautiful unsolved mathematical problems that can be shared with children. One that we shared in elementary classrooms is illustrated in Figure 7.

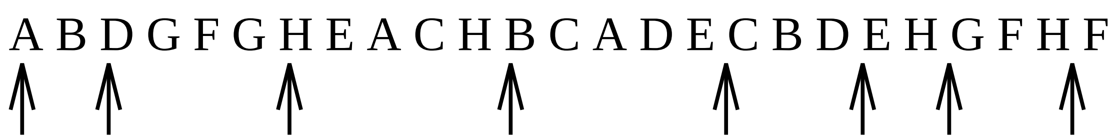

Figure 7: An open problem in discrete mathematics. If each letter appears exactly three times in a sequence of letters, it is possible to select one copy of each letter in such a way that no two selected letters are adjacent?

The conjecture is that if you create any sequence of letters (using an alphabet of any size), where each letter appears three times, it will always be possible to select one copy of each letter in such a way that no two selected copies are adjacent in the sequence. Figure 7 shows an example of such a sequence, where the alphabet is $\{A, ..., H\}$ (8 letters). Since each letter appears 3 times in the sequence, the sequence has length 24. The 8 arrows select one of each of the letters. Note that no two of the selected letters are adjacent in the sequence, i.e. no two of the arrows are immediately next to each other.

This conjecture remains unproved despite strenuous efforts by a number of mathematicians to settle it. There are many such beautiful and sometimes famous conjectures concerning combinatorial patterns and relationships that are accessible to young children and that can be manipulatively explored.

7.4 Communication

The tremendous importance of communication, and the fact that the bulk of one's time as a scientist is not spent in discovering things, but rather in communicating those discoveries, tends to be not widely appreciated outside of the science research communities. The traditional school mathematics classroom, with the emphasis on silent individual seat-work with very little writing, could hardly be more different in character. In the mathematics research community, many different modes of communication have a vital role, including electronic mail, formal, informal and very informal verbal communication, and carefully written archives. The classroom mathematics community can also be conducted in such a way that a variety of communication modes have an important role.

Participation in Larger Research Communities

As mentioned above, some mathematical problems in combinatorics could actually be investigated by elementary school children, although one might expect these opportunities to be somewhat limited. This is one way in which the classroom research community can be connected to larger research communities.

Another possible connection that a classroom can make is to communicate with mathematicians and computer scientists in the community, asking them for answers to questions, or inviting them to visit. [2]

The same could be done with any of the sciences. The mathematical sciences have a slight advantage in providing a model for classroom research communities in that their laboratories are so portable!

8 An Unexpected Windfall

It may be of interest to professional mathematicians to hear that the classroom experiences described in this paper led to some interesting research problems. Here are a few stories of such interaction.

1. There were staging difficulties with presenting the zero-knowledge protocol for small dominating sets with 7-year-olds (one of our more ambitious projects). Blum's algorithm requires quite a few envelopes, and these were, to put it bluntly, *incompetently numbered* (with transposed and illegible backwards numerals, etc.) by one of our assistants. The

[2] A touching example of this was the material given by one young student to a colleague to take to the MER workshop, "to show to the mathematicians and see what they think."

protocol is relatively sensitive to having a correctly labeled set of envelopes! But this raised an interesting question. How many envelopes do you actually need for a round that yields certainty at least 1/2? A research paper on this very topic appeared at about the same time.

2. The sorting network project was extremely popular and the kids requested a repeat. On the appointed day, I was very short of time and of my copy of Knuth. I thought I might quickly work out from scratch a sorting network for 8 inputs — but it was quite difficult! With the pressure on, it occurred to me that a certain randomized construction method might work. We tried that, and it did work. With hindsight, it was easy to make a heuristic argument for the success of the method, but a theorem would be more difficult. I mentioned the problem to a colleague shortly afterwards, who subsequently became involved in writing a paper on this subject [LP].

3. Last spring a colleague (Jan Kratochvil) visited for several months from Prague, and graciously came along on several of my school visits (which was easy, since he rented a room at the Apple Blossom Family School). After several of these sessions we spent some hours trying to solve problems that "turned up" on these adventures. For example, children quickly grasp the idea of a proper coloring of a map. A natural introduction for a child to possible strategies for minimizing the number of colors might involve taking turns coloring regions. The Four Color Theorem assures, if you are playing perfectly (and by yourself) that four colors are enough — but what if you alternate turns with "incompetent help" who make moves that are at least legal? An upper bound of 33 (probably not tight) was established by Kierstead and Trotter in 1992 [KT].

4. One of the projects with one-way functions involved creating graphs for which one knows a small dominating set, which can be very difficult (apparently!) for anyone else to find or match in size. But a theorem that would make this explicit must grapple with a thorny conundrum of complexity theory. Although, for example, there are more-than-polynomially many hard instances of an NP-complete problem such as Dominating Set (unless $P = NP$), it may be that the hard instances are still relatively rare, or difficult to generate. One wants a theorem (so it turns out) that P-sampleable distributions are hard for average case fixed-parameter search complexity, an interesting variation on a hot topic of current research in theoretical computer science.

In retrospect, why wouldn't one expect this sort of valuable feedback from the experience of explaining things to children? It would seem to be appropriate if contact between graduate students (who are finally emerging into the realm where they can *play again* at learning) and small children (who are living in world of play) to share and explain live science were a regular and normal part of our research lifestyle and training. We should make a loop of shared experience like this, and gradually draw it in, until all of schooling becomes language-learning play, connected to real, current live projects.

9 Acknowledgements.

Thanks especially to Marfa Levine, without whose persistent pursuit of a educational dream this adventure would perhaps not have materialized, and to Nancy Casey for inspiration and guidance in the ideas and literature of the whole language outlook on education. Thanks to Carole Moore and Prudy Heimsch for making their classrooms and experience available to us.

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